1.3. Prüfungsaufgaben zu Bruchgleichungen

Aufgabe 1: Lineare Bruchgleichungen ohne Variable im Nenner
Bestimme die Lösungsmenge der folgenden Gleichungen:

a) $\frac{3x+4}{8} + \frac{5x-4}{6} = \frac{2x-3}{4} + \frac{7}{12}$

b) $\frac{2x+3}{3} + \frac{2x-4}{5} = \frac{x-3}{6} + \frac{5}{2}$

Lösungen:

a) $\frac{3x+4}{8} + \frac{5x-4}{6} = \frac{2x-3}{4} + \frac{7}{12}$ $\quad | \cdot 24$

$9x + 12 + 20x - 16 = 12x - 18 + 14 \quad | -12x; +4$

$17x = 0$

$\Rightarrow L = \{0\}$

b) $\frac{2x+3}{3} + \frac{2x-4}{5} = \frac{x-3}{6} + \frac{5}{2}$ $\quad | \cdot 30$

$20x + 30 + 12x - 24 = 5x - 15 + 75 \quad | -5x; -6$

$27x = 54 \quad | :27$

$\Rightarrow L = \{2\}$

Aufgabe 2: Lineare Bruchgleichung mit Variable im Nenner ohne binomische Formeln

Bestimme Definitions– und Lösungsmenge der folgenden Gleichungen auf der Grundmenge G = $\mathbb{R}$:

a) $\frac{x-5}{x-2} = 1 - \frac{x+1}{x-2}$

b) $3 - \frac{x+2}{x-1} = \frac{x-4}{x-1}$

c) $\frac{2}{1+2x} = \frac{9}{3+6x} - \frac{1}{4-x}$

d) $\frac{1}{x-3} + \frac{3}{x} = \frac{6}{2x} + \frac{1}{2x-6}$

Lösungen

a) $\frac{x-5}{x-2} = 1 - \frac{x+1}{x-2}$ $\quad |\cdot(x-2)$ $\quad D = \mathbb{R}\backslash\{2\}$ (2)

$x - 5 = x - 2 - x - 1 \quad | +5$ (1)

$x = 2 \quad$ | mit Definitionsmenge vergleichen (1)

$\Rightarrow L = \{\}$ (1)

b) $3 - \frac{x+2}{x-1} = \frac{x-4}{x-1}$ $\quad |\cdot (x-1)$ $\quad D = \mathbb{R}\backslash\{1\}$, (2)

$3x - 3 - x - 2 = x - 4 \quad | -x; +5$ (1)

$x = 1 \quad$ | mit Definitionsmenge vergleichen (1)

$\Rightarrow L = \{\}$ (1)

c) $\frac{2}{1+2x} = \frac{9}{3+6x} - \frac{1}{4-x}$ $\quad |\cdot 3(1+2x)(4-x)$ $\quad D = \mathbb{R}\backslash\{-2; 4\}$ (2)

$24 - 6x = 36 - 9x - 3 - 6x \quad | + 9x; -24$ (1)

$9x = 9 \quad$ | :9 und mit Definitionsmenge vergleichen (1)

$\Rightarrow L = \{1\}$ (1)

d) $\frac{1}{x-3} + \frac{3}{x} = \frac{6}{2x} + \frac{1}{2x-6}$ $\quad |\cdot 2x(x-3)$ $\quad D = \mathbb{R}\backslash\{0; 3\}$ (2)

$2x + 6x - 18 = 6x - 18 + x \quad | -x$ (1)

$x = 0 \quad$ | mit Definitionsmenge vergleichen (1)

$\Rightarrow L = \{\}$ (1)

Aufgabe 3: Lineare Bruchgleichung mit Variable im Nenner und binomischen Formeln

Bestimme Definitions– und Lösungsmenge der folgenden Gleichungen auf der Grundmenge G = ℝ:

a) $\frac{5}{x+1} = \frac{8}{x} - \frac{3}{x-1}$

b) $\frac{5}{x} - \frac{3}{x+2} = \frac{2}{x-2}$

c) $\frac{4}{x^2-8x+16} = \frac{1}{x} - \frac{1}{x-4}$

d) $\frac{1}{x-3} - \frac{x}{x^2-9} = \frac{1}{(x+3)^2}$

e) $\frac{2}{x-4} - \frac{x-2}{x^2-8x+16} - \frac{1}{x} = 0$

f) $\frac{1}{x} - \frac{x}{x^2-6x+9} + \frac{5}{x^2-3x}$

g) $\frac{1}{x} - \frac{2}{x^2+6x+9} = \frac{x+1}{x^2+3x}$

h) $\frac{1}{x^2+2x} - \frac{2}{x^2-4} = \frac{1}{x^2-2x}$

i) $\frac{1}{x^2+3x} - \frac{2}{x^2-9} = \frac{1}{x^2-3x}$

j) $\frac{1}{x} - \frac{2}{x^2+6x+9} = \frac{x+1}{x^2+3x}$

k) $\frac{2x}{x^2-5x+6} - \frac{x+2}{x^2-3x} = \frac{x+3}{x^2-2x}$

l) $\frac{2}{x-5} - \frac{x+25}{x^2+10x+25} - \frac{1}{x+5} = 0$

m) $\frac{2}{x-3} - \frac{x+9}{x^2-9} - \frac{1}{x} = 0$

n) $\frac{2x+60}{x^2-25} - \frac{7}{x-5} = \frac{6}{x+5}$

o) $\frac{5}{2-5x} - \frac{12x+18}{4-25x^2} + \frac{4}{2+5x} = 0$

Lösungen:

a) $\frac{5}{x+1} = \frac{8}{x} - \frac{3}{x-1}$ | $\cdot\ x(x-1)(x+1)$

$5x(x-1) = 8(x-1)(x+1) - 3x(x+1)$ | ausmultiplizieren

$5x^2 - 5x = 8x^2 - 8 - 3x^2 - 3x$ | $-5x^2$; $+3x$

$-2x = -8$ | : (-2)

⇒ D = ℝ\{−1; 0; 1} und L = {4}

b) $\frac{5}{x} - \frac{3}{x+2} = \frac{2}{x-2}$ | $\cdot\ x(x-2)(x+2)$

$5x^2 - 20 - 3x^2 + 6x = 2x^2 + 4x$ | $-2x^2$; $-6x$

$-20 = -2x$ | : (-2)

⇒ D = ℝ\{−2; 0; 2} und L = {10}

c) $\frac{4}{x^2-8x+16} = \frac{1}{x} - \frac{1}{x-4}$ | $\cdot\ x(x-4)^2$

$4x = x^2 - 8x + 16 - x^2 + 4x$ | $-x^2$; $+4x$

$8x = 16$ | :8

⇒ D = ℝ\{0; 4} und, L = {2}

d) $\frac{1}{x-3} - \frac{x}{x^2-9} = \frac{1}{(x+3)^2}$ | $\cdot\ (x-3)(x+3)^2$

$x^2 + 6x + 9 - x^2 - 3x = x - 3$ | $-x$; -9

$2x = -12$ | :2

⇒ D = ℝ\{−3, 3} und L = {-6}

e) $$\frac{2}{x-4} - \frac{x-2}{x^2-8x+16} - \frac{1}{x} = 0 \qquad | \cdot x(x-4)^2$$
$$2x^2 - 8x - x^2 + 2x - x^2 + 8x - 16 = 0 \qquad | + 16$$
$$2x = 16 \qquad | :2$$

$\Rightarrow$ D = $\mathbb{R}\backslash\{0, 4\}$ und L = {8}

f) $$\frac{1}{x} - \frac{x}{x^2-6x+9} + \frac{5}{x^2-3x} = 0 \qquad | \cdot x(x-3)^2$$
$$x^2 - 6x + 9 - x^2 + 5x - 15 = 0 \qquad | + 6$$
$$-x = 6 \qquad | \cdot(-1)$$

$\Rightarrow$ D = $\mathbb{R}\backslash\{0; 3\}$ und L = {−6}

g) $$\frac{1}{x} - \frac{2}{x^2+6x+9} = \frac{x+1}{x^2+3x} \qquad |\cdot x(x+3)^2$$
$$x^2 + 6x + 9 - 2x = x^2 + 4x + 3 \qquad | -x^2; -4x$$
$$9 = 3$$

$\Rightarrow$ D = $\mathbb{R}\backslash\{-3; 0\}$ und L = {}

h) $$\frac{1}{x^2+2x} - \frac{2}{x^2-4} = \frac{1}{x^2-2x} \qquad | \cdot x(x+2)\cdot(x-2)$$
$$x - 2 - 2x = x + 2 \qquad | +x$$
$$-2x = 4$$

$\Rightarrow$ D = $\mathbb{R}\backslash\{-2, 0, 2\}$ und L = {}

i) $$\frac{1}{x^2+3x} - \frac{2}{x^2-9} = \frac{1}{x^2-3x} \qquad | \cdot x(x+3)(x-3)$$
$$x - 3 - 2x = x + 3 \qquad | - x; +3$$
$$-2x = 6 \qquad | :(-2)$$

$\Rightarrow$ D = $\mathbb{R}\backslash\{-3; 0; 3\}$ und L = {}

j) $$\frac{1}{x} - \frac{2}{x^2+6x+9} = \frac{x+1}{x^2+3x} \qquad |\cdot x(x+3)^2$$
$$x^2 + 6x + 9 - 2x = x^2 + 4x + 3 \qquad | -x^2; -4x$$
$$9 = 3$$

$\Rightarrow$ D = $\mathbb{R}\backslash\{-3; 0\}$ und L = {}

k) $$\frac{2x}{x^2-5x+6} - \frac{x+2}{x^2-3x} = \frac{x+3}{x^2-2x} \qquad | \cdot x(x-2)(x-3)$$
$$2x^2 - x^2 + 4 = x^2 - 9 \qquad | - x^2$$
$$4 = -9$$

$\Rightarrow$ D = $\mathbb{R}\backslash\{0; 2; 3{,}0\}$ und L = {}

l) $$\frac{2}{x-5} - \frac{x+25}{x^2+10x+25} - \frac{1}{x+5} = 0 \qquad | \cdot(x-5)(x+5)^2$$
$$2x^2 + 20x + 50 - x^2 - 20x + 125 - x^2 + 25 = 0 \qquad | \text{ zusammenfassen}$$
$$200 = 0$$

$\Rightarrow$ D = $\mathbb{Q}\backslash\{-5, 5\}$ und L = {}

m) $$\frac{2}{x-3} - \frac{x+9}{x^2-9} - \frac{1}{x} = 0 \qquad | \cdot x(x-3)(x+3)$$
$$2x^2 + 6x - x^2 - 9x - x^2 + 9 = 0 \qquad | + 3x$$
$$9 = 3x \qquad | :3$$

$\Rightarrow$ D = $\mathbb{R} \backslash \{-3, 0, 3\}$ und L = {}

n) $$\frac{2x+60}{x^2-25} - \frac{7}{x-5} = \frac{6}{x+5} \qquad | \cdot (x-5)(x+5)$$
$$2x + 60 - 7x - 35 = 6x - 30 \qquad | + 50; + 5x$$
$$55 = 11x \qquad | :11$$

$\Rightarrow$ D = $\mathbb{R}\backslash\{-5; 5\}$ und L = {}

o) $$\frac{5}{2-5x} - \frac{12x+18}{4-25x^2} + \frac{4}{2+5x} = 0 \qquad | \cdot(2-5x)(2+5x)$$
$$10 + 25x - 12x - 18 + 8 - 20x = 0$$
$$-7x = 0$$

D = $\mathbb{R} \backslash \{-\frac{2}{5}; \frac{2}{5}\}$ und L = {0}

Problem 4a: Equations including fractions and quadratic expressions (6)

a) Simplify the term $\frac{x-1}{x^2+x-12}-\frac{x-2}{x^2-6x+9}$ (4)

b) Solve to x the equation $\frac{x-1}{x^2+x-12}=\frac{x-2}{x^2-6x+9}$ (2)

Solutions:

a) $$\frac{x-1}{x^2+x-12}-\frac{x-2}{x^2-6x+9} = \frac{x-1}{(x+4)(x-3)}-\frac{x-2}{(x-3)^2} \quad (1)$$

$$= \frac{(x-1)(x-3)}{(x+4)(x-3)^2}-\frac{(x-2)(x+4)}{(x+4)(x-3)^2} \quad (1)$$

$$= \frac{x^2-4x+3-(x^2+2x-8)}{(x+4)(x-3)^2} \quad (1)$$

$$= \frac{-6x+11}{(x+4)(x-3)^2} \quad (1)$$

b) $\frac{x-1}{x^2+x-12}=\frac{x-2}{x^2-6x+9} \Leftrightarrow x^2 - 4x + 3 = x^2 + 2x - 8 \Leftrightarrow x = \frac{11}{6}$ (2)

Problem 4b: Equations including fractions and quadratic expressions (6)

Simplify $\frac{x+1}{x^2-3x-10}-\frac{x-2}{x^2-10x+25}$ (4)

Solve to x: $\frac{x}{x^2-3x-10}=\frac{1}{x^2-10x+25}$ (2)

Solutions:

a) $$\frac{x+1}{x^2-3x-10}-\frac{x-2}{x^2-10x+25} = \frac{x+1}{(x+2)(x-5)}-\frac{x-2}{(x-5)^2} \quad (1)$$

$$= \frac{(x+1)(x-5)}{(x+2)(x-5)^2}-\frac{(x-2)(x+2)}{(x+2)(x-5)^2} \quad (1)$$

$$= \frac{x^2-4x-5-(x^2-4)}{(x+2)(x-5)^2} \quad (1)$$

$$= \frac{-4x-1}{(x+2)(x-5)^2} \quad (1)$$

b) $\frac{x+1}{x^2-3x-10}=\frac{x-2}{x^2-10x+25} \Leftrightarrow x^2 - 4x - 5 = x^2 - 4 \Leftrightarrow x = -\frac{1}{4}$ (2)

Aufgabe 5: Lineare Bruchgleichungen mit Variable im Nenner und Parameter

Bestimme die Definitions- und die Lösungsmenge der folgenden Gleichung auf der Grundmenge $\mathbb{R}$ in Abhängigkeit vom Parameter $a \in \mathbb{R}$:

a) $\frac{2x-a}{4x^2+12x+9} = \frac{a-1}{4x+6}$ b) $\frac{x^2+a^2}{x^2-ax}-\frac{a^2+1}{ax-a^2} = 1$

Lösungen:

a) $\frac{2x-a}{4x^2+12x+9} = \frac{a-1}{4x+6}$ | $\cdot\ 2(2x+3)^2$ $D = \mathbb{R}\backslash\{-\frac{3}{2}\}$ (2)

$4x - 2a = 2ax - 2x + 3a - 3$ | $+2a; -2ax; +2x$ (1)

$(6 - 2a)x = 5a - 3$ | $:(6 - 2a)$ (1)

$\Rightarrow L = \{\frac{5a-3}{6-2a}\}$, falls $a \neq 6$ und $L = \{\}$, falls $a = 6$. (3)

b) $\frac{x^2+a^2}{x^2-ax}-\frac{a^2+1}{ax-a^2} = 1$ | $\cdot\ ax(x-a)$ $D = \mathbb{R}\backslash\{0; a\}$ (2)

$ax^2 + a^3 - a^2 x - x = ax^2 - a^2x$ | $-ax^2; +a^2x; + x$ (1)

$a^3 = x|$ (1)

$\Rightarrow L = \{a^3\}$. (3)